International J.Math. Combin. Vol.4 (2010), 46-52

The Upper Monophonic Number of a Graph

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Abstract: For a connected graph $G = (V, E)$, a Smarandachely k-monophonic set of G is a set $M \subseteq V(G)$ such that every vertex of G is contained in a path with less or equal k chords joining some pair of vertices in M. The Smarandachely k-monophonic number $m_S^k(G)$ of G is the minimum order of its Smarandachely k-monophonic sets. Particularly, a Smarandachely 0-monophonic path, a Smarandachely 0-monophonic number is abbreviated to a *monophonic path*, *monophonic number* $m(G)$ of G respectively. Any monophonic set of order $m(G)$ is a minimum monophonic set of G. A monophonic set M in a connected graph G is called a minimal monophonic set if no proper subset of M is a monophonic set of G. The upper monophonic number $m^+(G)$ of G is the maximum cardinality of a minimal monophonic set of G. Connected graphs of order p with upper monophonic number p and $p-1$ are characterized. It is shown that for every two integers a and b such that $2 \leq a \leq b$, there exists a connected graph G with $m(G) = a$ and $m^+(G) = b$.

Key Words: Smarandachely k-monophonic path, Smarandachely k-monophonic number, monophonic path, monophonic number.

AMS(2010): 05C12

§1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [1]. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ *geodesic*. A vertex x is said to *lie on* a $u - v$ geodesic P if x is a vertex of P including the vertices u and v. The *eccentricity* $e(v)$ of a vertex v in G is the maximum distance from v and a vertex of G. The minimum eccentricity among the vertices of G is the *radius*, *rad* G or $r(G)$ and the maximum eccentricity is its *diameter*, *diam* G of G. A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices of S. The *geodetic number* $g(G)$ of G is the minimum cardinality of its geodetic sets and any

[1]Received August 6, 2010. Accepted December 16, 2010.

geodetic set of cardinality $g(G)$ is a *minimum geodetic set* of G. The geodetic number of a graph is introduced in [2] and further studied in [3]. $N(v) = \{u \in V(G) : uv \in E(G)\}$ is called the *neighborhood* of the vertex v in G. For any set S of vertices of G, the *induced subgraph* $< S >$ is the maximal subgraph of G with vertex set S. A vertex v is an *extreme vertex* of a graph G if $< N(v) >$ is complete. A *chord* of a path $u_0, u_1, u_2, \ldots, u_h$ is an edge u_iu_j, with $j \geq i + 2$. An $u - v$ path is called a *monophonic path* if it is a chordless path. A Smarandachely k-monophonic set of G is a set $M \subseteq V(G)$ such that every vertex of G is contained in a path with less or equal k chords joining some pair of vertices in M. The Smarandachely k-monophonic number $m_S^k(G)$ of G is the minimum order of its Smarandachely k-monophonic sets. Particularly, a Smarandachely 0-monophonic path, a Smarandachely 0-monophonic number is abbrevated to *monophonic path*, *monophonic number* $m(G)$ of G respectively. Thus, a *monophonic set* of G is a set $M \subseteq V$ such that every vertex of G is contained in a monophonic path joining some pair of vertices in M. The monophonic number $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a *minimum monophonic set* or simply a $m - set$ of G. It is easily observed that no cut vertex of G belongs to any minimum monophonic set of G. The monophonic number of a graph is studied in [4, 5, 6]. For the graph G given in Figure 1.1, $S_1 = \{v_2, v_4, v_5\}$, $S_2 = \{v_2, v_4, v_6\}$ are the only minimum geodetic sets of G so that $g(G) = 3$. Also, $M_1 = \{v_2, v_4\}, M_2 = \{v_4, v_6\}, M_3 = \{v_2, v_5\}$ are are the only minimum monophonic sets of G so that $m(G) = 2$.

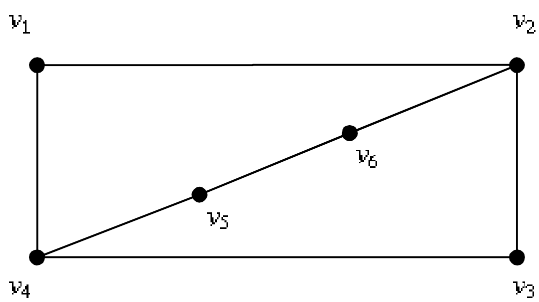

Figure 1: G

§2. The Upper Monophonic Number of a Graph

Definition 2.1 *A monophonic set M in a connected graph G is called a* minimal monophonic set *if no proper subset of M is a monophonic set of G. The upper monophonic number $m^+(G)$ of G is the maximum cardinality of a minimal monophonic set of G.*

Example 2.2 For the graph G given in Figure 1.1, $M_4 = \{v_1, v_3, v_5\}$ and $M_5 = \{v_1, v_3, v_6\}$ are minimal monophonic sets of G so that $m^+(G) \geq 3$. It is easily verified that no four element subsets or five element subsets of $V(G)$ is a minimal monophonic set of G and so $m^+(G) = 3$.

Remark 2.3 Every minimum monophonic set of G is a minimal monophonic set of G and the converse is not true. For the graph G given in Figure 1.1, $M_4 = \{v_1, v_3, v_5\}$ is a minimal monophonic set but not a minimum monophonic set of G.

Theorem 2.4 *Each extreme vertex of G belongs to every monophonic set of G.*

Proof Let M be a monophonic set of G and v be an extreme vertex of G. Let $\{v_1, v_2, \ldots, v_k\}$ be the neighbors of v in G. Suppose that $v \notin M$. Then v lies on a monophonic path $P : x = x_1, x_2, \ldots, v_i, v, v_j, \ldots, x_m = y$, where $x, y \in M$. Since v_iv_j is a chord of P and so P is not a monophonic path, which is a contradiction. Hence it follows that $v \in M$. $\square$

Theorem 2.5 *Let G be a connected graph with cut-vertices and S be a monophonic set of G. If v is a cut -vertex of G, then every component of $G - v$ contains an element of S.*

Proof Suppose that there is a component G_1 of $G - v$ such that G_1 contains no vertex of S. By Theorem 2.4, G_1 does not contain any end-vertex of G. Thus G_1 contains at least one vertex, say u. Since S is a monophonic set, there exists vertices $x, y \in S$ such that u lies on the $x - y$ monophonic path $P : x = u_0, u_1, u_2, \ldots, u, \ldots, u_t = y$ in G. Let P_1 be a $x - u$ sub path of P and P_2 be a $u - y$ subpath of P. Since v is a cut-vertex of G, both P_1 and P_2 contain v so that P is not a path, which is a contradiction. Thus every component of $G - v$ contains an element of S. $\square$

Theorem 2.6 *For any connected graph G, no cut-vertex of G belongs to any minimal monophonic set of G.*

Proof Let M be a minimal monophonic set of G and $v \in M$ be any vertex. We claim that v is not a cut vertex of G. Suppose that v is a cut vertex of G. Let $G_1, G_2, \ldots, G_r (r \geq 2)$ be the components of $G - v$. By Theorem 2.5, each component $G_i (1 \leq i \leq r)$ contains an element of M. We claim that $M_1 = M - \{v\}$ is also a monophonic set of G. Let x be a vertex of G. Since M is a monophonic set, x lies on a monophonic path P joining a pair of vertices u and v of M. Assume without loss of generality that $u \in G_1$. Since v is adjacent to at least one vertex of each $G_i (1 \leq i \leq r)$, assume that v is adjacent to z in G_k, $k \neq 1$. Since M is a monophonic set, z lies on a monophonic path Q joining v and a vertex w of M such that w must necessarily belongs to G_k. Thus $w \neq v$. Now, since v is a cut vertex of G, $P \cup Q$ is a path joining u and w in M and thus the vertex x lies on this monophonic path joining two vertices u and w of M_1. Thus we have proved that every vertex that lies on a monophonic path joining a pair of vertices u and v of M also lies on a monophonic path joining two vertices of M_1. Hence it follows that every vertex of G lies on a monophonic path joining two vertices of M_1, which shows that M_1 is a monophonic set of G. Since $M_1 \subsetneq M$, this contradicts the fact that M is a minimal monophonic set of G. Hence $v \notin M$ so that no cut vertex of G belongs to any minimal monophonic set of G. $\square$

Corollary 2.7 *For any non-trivial tree T, the monophonic number $m^+(T) = m(T) = k$, where k is number of end vertices of T.*

Proof This follows from Theorems 2.4 and 2.6. $\square$

Corollary 2.8 *For the complete graph $K_p(p \geq 2)$, $m^+(K_p) = m(K_p) = p$.*

Proof Since every vertex of the complete graph, $K_p(p \geq 2)$ is an extreme vertex, the vertex set of K_p is the unique monophonic set of K_p. Thus $m^+(K_p) = m(K_p) = p$. □

Theorem 2.9 *For a cycle $G = C_p(p \geq 4)$, $m^+(G) = 2 = m(G)$.*

Proof Let x, y be two independent vertices of G. Then $M = \{x, y\}$ is a monophonic set of G so that $m(G) = 2$. We show that $m^+(G) = 2$. Suppose that $m^+(G) > 2$. Then there exists a minimal monophonic set M_1 such that $|M_1| \geq 3$. Now it is clear that $M \subsetneq M_1$, which is a contradiction to M_1 a minimal monophonic set of G. Therefore, $m^+(G) = 2$. □

Theorem 2.10 *For a connected graph G, $2 \leq m(G) \leq m^+(G) \leq p$.*

Proof Any monophonic set needs at least two vertices and so $m(G) \geq 2$. Since every minimal monophonic set is a monophonic set, $m(G) \leq m^+(G)$. Also, since $V(G)$ is a monophonic set of G, it is clear that $m^+(G) \leq p$. Thus $2 \leq m(G) \leq m^+(G) \leq p$. □

The following Theorem is proved in [3].

Theorem A *Let G be a connected graph with diameter d .Then $g(G) \leq p - d + 1$.*

Theorem 2.11 *Let G be a connected graph with diameter d .Then $m(G) \leq p - d + 1$.*

Proof Since every geodetic set of G is a monophonic set of G, the assertion follows from Theorem 2.10 and Theorem A. □

Theorem 2.12 *For a non-complete connected graph G, $m(G) \leq p - k(G)$, where $k(G)$ is vertex connectivity of G.*

Proof Since G is non complete, it is clear that $1 \leq k(G) \leq p - 2$. Let $U = \{u_1, u_2, \ldots, u_k\}$ be a minimum cutset of vertices of$G$. Let $G_1, G_2, \ldots, G_r(r \geq 2)$ be the components of $G - U$ and let $M = V(G) - U$. Then every vertex $u_i(1 \leq i \leq k)$ is adjacent to at least one vertex of $G_j(1 \leq j \leq r)$. Then it follows that the vertex u_i lies on the monophonic path x, u_i, y, where $x, y \in M$ so that M is a monophonic set. Thus $m(G) \leq p - k(G)$. □

The following Theorems 2.13 and 2.15 characterize graphs for which $m^+(G) = p$ and $m^+(G) = p - 1$ respectively.

Theorem 2.13 *For a connected graph G of order p, the following are equivalent:*

(i) $m^+(G) = p$;
(ii) $m(G) = p$;
(iii) $G = K_p$.

Proof $(i) \Rightarrow (ii)$. Let $m^+(G) = p$. Then $M = V(G)$ is the unique minimal monophonic set of G. Since no proper subset of M is a monophonic set, it is clear that M is the unique minimum monophonic set of G and so $m(G) = p$. $(ii) \Rightarrow (iii)$. Let $m(G) = p$. If $G \neq K_p$, then by Theorem 2.11, $m(G) \leq p-1$, which is a contradiction. Therefore $G = K_p$. $(ii) \Rightarrow (iii)$. Let $G = K_p$. Then by Corollary 2.8, $m^+(G) = p$. $\square$

Theorem 2.14 *Let G be a non complete connected graph without cut vertices. Then $m^+(G) \leq p-2$.*

Proof Suppose that $m^+(G) \geq p-1$. Then by Theorem 2.13, $m^+(G) = p-1$. Let v be a vertex of G and let $M = V(G) - \{v\}$ be a minimal monophonic set of G. By Theorem 2.4, v is not an extreme vertex of G. Then there exists $x, y \in N(v)$ such that $xy \notin E(G)$. Since v is not a cut vertex of G, $< G - v >$ is connected. Let $x, x_1, x_2, \ldots, x_n, y$ be a monophonic path in $< G - v >$. Then $M_1 = M - \{x_1, x_2, \ldots, x_n\}$ is a monophonic set of G. Since $M_1 \subsetneq M$, M_1 is not a minimal monophonic set of G, which is a contradiction. Therefore $m^+(G) \leq p-2$. $\square$

Theorem 2.15 *For a connected graph G of order p, the following are equivalent:*

(i) $m^+(G) = p-1$;
(ii) $m(G) = p-1$;
(iii) $G = K_1 + \bigcup m_j K_j$, $\sum m_j \geq 2$.

Proof $(i) \Rightarrow (ii)$. Let $m^+(G) = p-1$. Then it follows from Theorem 2.13 that G is non-complete. Hence by Theorem 2.14, G contains a cut vertex, say v. Since $m^+(G) = p-1$, hence it follows from Theorem 2.6 that $M = V - \{v\}$ is the unique minimal monophonic set of G. We claim that $m(G) = p-1$. Suppose that $m(G) < p-1$.Then there exists a minimum monophonic set M_1 such that $|M_1| < p-1$. It is clear that $v \notin M_1$. Then it follows that $M_1 \subsetneq M$, which is a contradiction. Therefore $m(G) = p-1$. $(ii) \Rightarrow (iii)$. Let $m(G) = p-1$. Then by Theorem 2.11, $d \leq 2$. If $d = 1$, then $G = K_p$, which is a contradiction. Therefore $d = 2$. If G has no cut vertex, then by Theorem 2.12, $m(G) \leq p-2$, which is a contradiction. Therefore G has a unique cut-vertex, say v. Suppose that $G \neq K_1 + \bigcup m_j K_j$. Then there exists a component, say G_1 of $G - v$ such that $< G_1 >$ is non complete. Hence $|V(G_1)| \geq 3$. Therefore $< G_1 >$ contains a chordless path P of length at least two. Let y be an internal vertex of the path P and let $M = V(G) - \{v, y\}$. Then M is a monophonic set of G so that $m(G) \leq p-2$, which is a contradiction. Thus $G = K_1 + \bigcup m_j K_j$. $(iii) \Rightarrow (i))$. Let $G = K_1 + \bigcup m_j K_j$. Then by Theorems 2.4 and 2.6, $m^+(G) = p-1$. $\square$

In the view of Theorem 2.10, we have the following realization result.

Theorem 2.16 *For any positive integers $2 \leq a \leq b$, there exists a connected graph G such that $m(G) = a$ and $m^+(G) = b$.*

Proof Let G be a graph given in Figure 2.1 obtained from the path on three vertices $P : u_1, u_2, u_3$ by adding the new vertices $v_1, v_2, \ldots, v_{b-a+1}$ and $w_1, w_2, \ldots, w_{a-1}$ and joining each v_i $(1 \leq i \leq b-a+1)$ to each $v_j (1 \leq j \leq b-a+1)$, $i \neq j$, and also joining each w_i $(1 \leq i \leq a-1)$ with u_1 and u_2. First we show that $m(G) = a$. Let M be a monophonic set of G and let $W = \{w_1, w_2, \ldots, w_{a-1}\}$. By Theorem 2.4, $W \subseteq M$. It is easily seen that W is not a monophonic set of G. However, $W \cup \{u_3\}$ is a monophonic set of G and so $m(G) = a$. Next we show that $m^+(G) = b$. Let $M_1 = W \cup \{v_1, v_2, \ldots, v_{b-a+1}\}$. Then M_1 is a monophonic

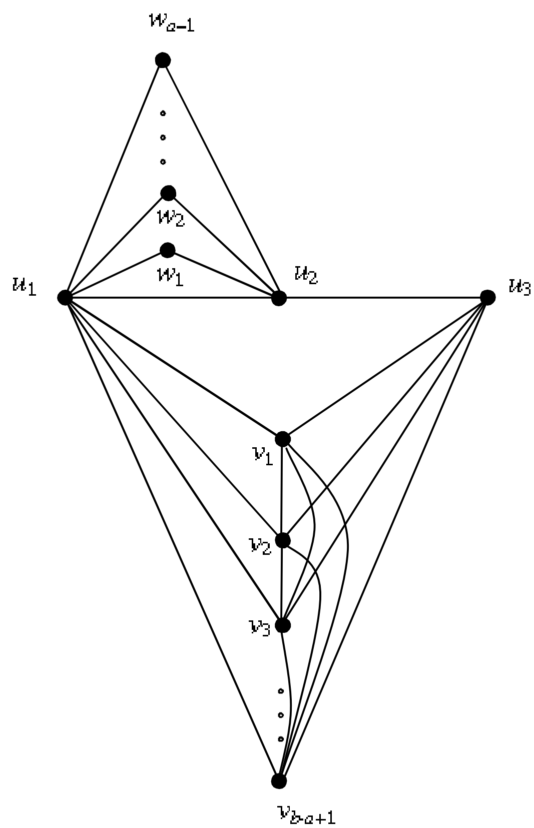

Figure 2: G

set of G. If M_1 is not a minimal monophonic set of G, then there is a proper subset T of M_1 such that T is a monophonic set of G. Then there exists $v \in M_1$ such that $v \notin T$. By Theorem 2.4, $v \neq w_i$ $(1 \leq i \leq a-1)$. Therefore $v = v_i$ for some i $(1 \leq i \leq b-a+1)$. Since $v_iv_j(1 \leq i, j \leq b-a+1)$, $i \neq j$ is a chord, v_i does not lie on a monophonic path joining some vertices of T and so T is not a monophonic set of G, which is a contradiction. Thus M_1 is a minimal monophonic set of G and so $m^+(G) \geq b$. Let T' be a minimal monophonic set of G with $|T'| \geq b+1$. By Theorem2.4, $W \subseteq T'$. Since $W \cup \{u_3\}$ is a monophonic set of G, $u_3 \notin T'$. Since M_1 is a monophonic set of G, there exists at least one v_i such that $v_i \notin T'$. Without loss of generality let us assume that $v_1 \notin T'$. Since $|T'| \geq b+1$, then u_1, u_2 must belong to T'. Now it is clear that v_1 does not lie on a monophonic path joining a pair of vertices of T', it follows that T' is not a monophonic set of G, which is a contradiction. Therefore $m^+(G) = b$. □

References

[1] F. Buckley and F. Harary, *Distance in Graphs*, Addition- Wesley, Redwood City, CA, 1990.

[2] F. Buckley, F. Harary, L. V. Quintas, Extremal results on the geodetic number of a graph, *Scientia A2*, (1988), 17-26.

[3] G. Chartrand, F. Harary, Zhang, On the Geodetic Number of a graph, *Networks*, 39(1),

(2002) 1 - 6.

[4] Carmen Hernando, Tao Jiang, Merce Mora, Ignacio. M. Pelayo and Carlos Seara, On the Steiner, geodetic and hull number of graphs, *Discrete Mathematics*, 293(2005), 139 - 154.

[5] Esamel M. paluga, Sergio R. Canoy, Jr. , Monophonic numbers of the join and Composition of connected graphs, *Discrete Mathematics*, 307 (2007), 1146 - 1154.

[6] Mitre C. Dourado, Fabio Protti and Jayme. L. Szwarcfiter, Algorithmic Aspects of Monophonic Convexity, *Electronic Notes in Discrete Mathematics*, 30(2008), 177-1822.